PHYSICAL REVIEW VOLUME 136, NUMBER 4B 23 NOVEMBER 1964

Analyticity of Field Theory*

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(Received 26 June 1964)

We prove that a generalized field theory has the same minimum surfaces of singularities in the external momenta as Chew's S matrix, given by all the possible Landau surfaces of singularity of the related perturbation theory. This implies in particular that Lagrangian field theories are a special class of models satisfying the S matrix postulates.

THERE seems to be three different frameworks with which to attempt to understand the elementary particles: (1) axiomatic field theory,[1] (2) S-matrix theory,[2] (3) Lagrangian field theory. Only the latter has definite equations, though many people regard these equations as inconsistent or meaningless.[3] We have extended (3) as far as possible by complete unitarity (C.U.)[4]; we can obtain (3) as a very simple type of C.U. equation. We want to consider here the relation between C.U. and S-matrix theory. In particular we want to show that C.U. has the same maximal analyticity as S-matrix theory independent of perturbation theory. It will be useful first to discuss the framework of C.U. before we go on to consider its analyticity. C.U. gives a method of writing down contributions to S-matrix elements which have up to a certain number of intermediate particles in a certain channel absent. These S-matrix elements are, in general, off the mass shell since the intermediate particles removed are off the mass shell. The contributions lacking intermediate particles are called irreducible with respect to these particles. C.U. means essentially that any S-matrix amplitude with n external particles is the sum of all possible amplitudes for all possible processes with any number of internal particles, but only n external particles. We consider, for simplicity, only one type of neutral pseudoscalar particle without loss of generality. Thus, we may apply C.U., for example, to the 2-particle scattering amplitude M, which is then the sum of all amplitudes with more than two internal particles in a given channel M_2 plus the sum of all amplitudes with a 2-particle intermediate state. This latter contribution will be

$$\int M(1256)D(5)D(6)\delta^4(1+2-5-6)\times M_2(5634)d^4p_5d^4p_6\,, \quad (1)$$

where p_1, p_2 are the initial momenta, p_3, p_4 final momenta, p_5, p_6 intermediate momenta of the particles, and $D(p)$ the one-particle propagator. Then

$$M=M_2+MKM_2\,, \quad (2)$$

where MKM_2 represents the product given by (1). We may regard (2) as a definition of M_2 from M, or vice versa. We say M_2 is 2-particle irreducible in the (1,2) channel. We can similarly define other irreducible amplitudes. So far C.U. is vacuous in that it gives a means of defining things we do not known—irreducible functions, from things we also do not know—S-matrix amplitudes off the mass shell. Its usefulness becomes apparent when we consider how Lagrangian field theory is related to C.U. If we consider a simple Lagrangian field theory, e.g., a scalar field theory with interaction $g\phi^3$, then in any S-matrix element every external particle is always joined by two lines to the rest of the amplitude. Thus, if $M_2(p;q_1\cdots q_n)$ is the irreducible amplitude with no two particles between the external particle with momentum p and the others with momenta $q_1\cdots q_n$, with

$$p=\sum_{i=1}^{n} q_i\,,$$

then

$$M_2(p;q_1\cdots q_n)=0 \quad (n>2)$$
$$=g \quad (n=2)\,. \quad (3)$$

It is also possible to show that[4]

$$M_2(p;p)=(p^2-m^2)\,. \quad (4)$$

The resulting C.U. equations with restrictions (3), (4) are identical to those for the amputated Green's func-

* The research reported in this document has been sponsored in part by the Air Force Office of Scientific Research, OAR, under Grant No. AF EOAR 63-79 with the European Office of Aerospace Research, U. S. Air Force.

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[1] A brief account of this approach and its relation to the S-matrix theory is given by R. Jost, in *Proceedings of the Sienna International Conference on Elementary Particles, 1963* (Italian Physical Society, Bologna), Vol. II, p. 140. We will not discuss axiomatic field theory here.

[2] G. F. Chew, *S-matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961). For more recent work, see, for example, P. V. Landshoff, Cambridge University (unpublished).

[3] G. F. Chew, Ref. 2, p. 1; Lawrence Radiation Laboratory Report 10891 (unpublished). F. Low, *Proceedings of the Sienna International Conference on Elementary Particles, 1963* (Italian Physical Society, Bologna), Vol. II, p. 137.

[4] J. G. Taylor, Nuovo Cimento, Suppl. (to be published). We use C. U. as an abbreviation for "complete unitarity."

tions arising from the $g\phi^3$ interaction.[5] Since the restrictions (3), (4) are very simple, and we have a definite set of possibly meaningless equations, we can now go ahead and see how meaningless they are. As to be expected there are certain technical difficulties in such a discussion—the equations are nonlinear, there are an infinite number of them, they involve principal value singularities and the functions involved are invariant under a noncompact group. The first two difficulties can be shown to be surmountable by existing mathematical techniques.[6] The remaining two difficulties are connected with the Lorentz metric of the space-time we are dealing with. The best way of avoiding these difficulties is to consider a Euclidean metric. This gives rise to a problem of analytic continuation on going back to the Lorentz metric so that a discussion of the analyticity of the S-matrix element is unavoidable. Without enough analyticity to perform this continuation, it would seem very difficult to discuss the equations. We would like to have as much analyticity as possible. How much can we have—what is the region of maximal analyticity, off the mass shell, for the possible solutions of C.U.? This has already been answered for the S-matrix theory.[7] The answer given there is that the smallest region of singularities consistent with unitarity is that given by all possible Landau surfaces of singularity for perturbation theory,[8] on the mass shell. We wish to determine the maximal region of analyticity arising in C.U.; in particular, we wish to prove the following theorem:

Theorem: The smallest region of singularities for any solution of the C.U. equations (with certain restrictions on the irreducible functions) is that given by all possible Landau surfaces of singularity of perturbation theory. This theorem will have as an immediate corollary, the following:

Corollary: The smallest region of singularities for any solution of the field equations arising from some local interaction is that given by all possible Landau surfaces of singularity of the perturbation theory corresponding to that interaction.

In order to prove the theorem, we need to consider the restrictions we must apply to the irreducible functions. We have remarked already that the C.U. equations are vacuous, only giving a definition of the irreducible functions from the S-matrix elements. We hope that enough of these irreducible functions are given to determine the S-matrix elements completely from the C.U. equations. This is to be expected, if, for example, all the irreducible functions $M_r(p;\, q_1, \cdots, q_n)$ are given for all $n \geqslant 1$ for a fixed r. Such a restriction can be regarded as arising if the S-matrix elements are the Green's functions of a field operator with some polynomial self interaction energy. This interaction may be nonlocal and will be of degree $r+1$.[8a]

We assume some set $M_r(p_1 \cdots p_n;\, q_1 \cdots q_m)$, given for suitable choices of r, n, and m (where M_r lacks up to r internal particles between the external particles with momenta p and those with momenta q). We assume further that each of the functions $M_r(p_1 \cdots p_n;\, q_1 \cdots q_n)$ only have singularities on the possible Landau surfaces of the corresponding perturbation theory. By the corresponding perturbation theory, we mean that arising from Feynman diagrams with vertices with $(r+1)$ lines meeting, r taking the values for the given M_r. Finally, we assume that the complete propagator has a single pole at $p^2 - m^2$.

We now prove the theorem under the above restriction on the irreducible function M_r. We regard the C.U. equations as defining a nonlinear mapping on the set of unknown amplitudes and irreducible functions into itself. For example, the way any line is joined through two lines is of the form

$$M(p;\, q_1 \cdots q_n) = \int M(p, r_1, r_2) D(r_1) D(r_2) M_2(r_1 r_2;\, q_1 \cdots q_n) \times \delta(r_1 + r_2 - p)\, dr_1 dr_2 + M_2(p;\, q_1 \cdots q_n). \quad (5)$$

Then, if $M_2(p;\, q_1 \cdots q_n)$ is given for all n, the right-hand side of (5) maps the unknown M's and M_2's nonlinearly into the M's on the left-hand side. The proof of our theorem will consist in showing that if we assume only Landau singularities in the functions on the right-hand side of (5) and similar equations then the left-hand side will only have these Landau singularities. To do that we integrate out all the δ functions in the nonlinear terms. The general structure of each of these nonlinear terms is the same as that of perturbation theory, where the lines are complete propagators, some of the vertices are unknown amplitudes with only Landau singularities, the remaining vertices are the given set of irreducible functions M_r, again with only Landau singularities. Then we would intuitively expect that only Landau surfaces of singularity can result from such perturbation-type nonlinear terms, and this follows immediately from the lemma[7]:

Lemma: The singularities of

$$F(z_k) = \int f(x_j, z_k) \prod_j dx_j$$

when the singularities of $f(x_j, z_k)$ lie on surfaces $S_i(x_j, z_k)$

[5] K. Symanzik, Report in Hercegnovi Summer School, Yugoslavia, 1959 (unpublished).

[6] J. G. Taylor, (to be published).

[7] J. C. Polkinghorne, Nuovo Cimento **23**, 360 (1962); **25**, 901 (1962); H. Stapp, Phys. Rev. **125**, 2139 (1962).

[8] L. Landau, Nucl. Phys. **13**, 181 (1959).

[8a] *Note added in proof.* This imposes no restriction on whether the particles are elementary or composite, since the composite particle case may be expressed in C.U. form with suitable restrictions on the coupling constant and wave-function renormalization constant for the composite particle. This has been discussed in Ref. 4 and in a forthcoming paper by the author.

$=0$ $(i=1,\cdots,n)$, are given by

$$\lambda_i S_i=0 \quad i=1,\cdots,n$$

$$\sum_{i=1}^{n}\lambda_i\frac{\partial S_i}{\partial x_j}=0 \quad j=1,2\cdots,$$

where the λ_i are a set of Lagrange multipliers.

This lemma may be proved intuitively by considering the function

$$\int\prod dx_j[\prod_i S_i(x_j z_k)]^{-1}$$

which may be written

$$\int\prod dx_j\int\prod d\lambda_i\delta(1-\sum\lambda_i)[\sum\lambda_i S_i]^{-n},$$

so will have the given singular surfaces. This is not the most general function satisfying the conditions of the lemma, so does not constitute a true proof. This may be given.[9] The lemma shows that the singularities of each of the nonlinear terms we are considering depend only on the singularities of the integrands. These are the same as in perturbation theory, so the perturbation theory discussion of Landau will go through for our case also, giving only Landau surfaces of singularity. This proves our theorem.

Our theorem essentially says that iteration of the C.U. equations preserves Landau singularities. We have not proved that this iteration will converge, and to a solution of the C.U. equations with only Landau singularities. This is an essential step in discussing the meaninglessness of field equations. We will report on this elsewhere.[6,9a]

Our theorem does not show that S-matrix theories and restricted C.U. theories are identical since there may exist solutions of C.U. without maximal analyticity. It does show that field theory, and more generally C.U. with certain restrictions (to ensure unitarity), gives a class of models satisfying the S-matrix conditions. These models have definite equations. It is interesting to see if the set of all these models, with all possible restrictions on C.U. to ensure unitarity, exhausts all possible solutions of the S-matrix program. Since the given irreducible functions seem to correspond to subtraction constants, we conjecture that this is the case. It seems then that C.U. enables us to give an over-all framework embracing both field theory and S-matrix theory, but being more general than both.

I would like to thank D. Branson and P. V. Landshoff and J. C. Polkinghorne for helpful discussions.

[9] P. Lelong, "Lecons sur la Theorie des Fonctions de Plusieurs Variables Complex," Saclay, 1960 (unpublished), Chap. 4.

[9a] *Note added in proof.* C.U. ensures that we may write the nonlinear mapping, used in proving our theorem, in such a way that at each step of its iteration the new value for each irreducible function only has branch points at the positions expected from the absence of certain irreducible particles. Thus the iteration of the equation for $M_r(p_1\cdots p_n;q_1\cdots q_m)$ does not reduce the lowest branch point in the variable $(p_1+\cdots+p_n)^2$ below $(r+1)^2m^2$.